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SASAKIAN MANIFOLDS AND M-THEORY

JOSÉ FIGUEROA-O'FARRILL, ANDREA SANTI

ABSTRACT. We extend the link between Einstein Sasakian manifolds and Killing spinors to a class of η -Einstein Sasakian manifolds, both in Riemannian and Lorentzian settings, characterising them in terms of generalised Killing spinors. We propose a definition of supersymmetric M-theory backgrounds on such a geometry and find a new class of such backgrounds, extending previous work of Haupt, Lukas and Stelle.

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1. INTRODUCTION

Sasakian manifolds (see, e.g., [1]) continue to play an important rôle in mathematical physics, ever since the emergence, almost two decades ago, of the conjectural gauge/gravity correspondence [2]. Klebanov and Witten [3] (following from earlier work of Kehagias [4]) conjectured that the gravity dual of a certain 4-dimensional $N = 1$ superconformal field theory was given by type IIB superstring theory on the product of 5-dimensional anti-de Sitter spacetime and a homogeneous five-dimensional Sasaki–Einstein manifold called $T^{1,1}$. They interpreted this ten-dimensional Lorentzian manifold as the near-horizon geometry of a stack of D3-branes sitting at the singularity of the conifold.

This interpretation was further explored and extended in [5, 6, 7], setting up a correspondence between superconformal field theories with less

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than maximal supersymmetry and near-horizon geometries of supersymmetric brane configurations, where the branes are located at a conical singularity in a Riemannian manifold of special holonomy. The near-horizon geometry of such branes is then metrically a product of an anti-de Sitter spacetime with the link of the cone, which is an Einstein manifold (or, more generally, an orbifold) admitting real Killing spinors. In particular, conical singularities of Calabi–Yau manifolds have links which are Sasakian manifolds. Indeed, one of the equivalent characterisations of a Sasakian manifold is one whose metric cone is Kähler and if, in addition, the Sasakian manifold is Einstein with positive scalar curvature, then the cone is Ricci-flat and hence Calabi–Yau. This is one instance of Bär’s cone construction [8], which states that the metric cone of an Einstein manifold admitting real Killing spinors is either flat or irreducible and admits parallel spinors.

Although the gauge/gravity correspondence exists between superconformal field theories and string/M-theory, it is the ’t Hooft (or large N) limit that has been studied the most, since that limit corresponds to the supergravity limit of the string or M-theory. It is believed that supersymmetric supergravity backgrounds which are dual to the large N limit of a superconformal field theory can be corrected (in a way analogous to the $1/N$ corrections of the field theory) to yield exact string/M-theory backgrounds, but in the case of M-theory this is hindered by the lack of a good working definition of the notion of a “supersymmetric M-theory background”.

The supergravity limit of M-theory is eleven-dimensional supergravity, the unique eleven-dimensional supergravity theory with 32 supercharges predicted by Nahm [9] and constructed by Cremmer, Julia and Scherk [10]. The first-order corrections to the Maxwell equations of eleven-dimensional supergravity were found in [11] by Duff, Liu and Minasian. They are often called the Green–Schwarz corrections and are needed by demanding the cancellation of anomalies in the worldvolume theory of the fivebrane. As we will review below, the Green–Schwarz term takes the form of a correction to the Chern–Simons term by adding to $F \wedge F$ an 8-form made out of the first and second Pontryagin forms.

What is still unclear are the corrections to the spinor connection defined from the supersymmetry variation of the gravitino. This connection encodes the geometry of the supersymmetric supergravity backgrounds: not just they define the notion of Killing spinor, but the bosonic field equations are equivalent to the vanishing of the gamma-trace of its curvature [12]. Due to the incomplete knowledge of the corrections to this connection, we lack the notion of a supersymmetric M-theory background (even to first order).

This motivates the search for eleven-dimensional Lorentzian geometries which admit spinor fields which are parallel relative to connections which are “close” (in a sense which is made precise in Section 4.2) to the connection in eleven-dimensional supergravity. In this paper we explore Lorentzian Sasakian manifolds admitting such spinor fields and obtain two main results.

The first is a characterisation of Sasakian manifolds, both in Riemannian and Lorentzian settings, whose transverse geometry is Ricci flat as those admitting nonzero “generalised Killing spinors”, a notion which is given in Definition 3.1. This generalises the characterisation of Sasaki–Einstein manifolds as those admitting Killing spinors. The second result is showing that a certain class of Lorentzian Sasakian spin manifolds which can be exhibited as bundles over Calabi–Yau 5-folds admit generalised Killing spinors and solve the first-order M-theory bosonic equations with a nonzero flux F .

We remark that these bundles are in general nontrivial and that the Riemannian metric of the Calabi–Yau 5-fold is essentially supported over the maximally non-integrable distribution which is naturally associated with the Sasakian manifold. This class of backgrounds is therefore complementary to previous solutions found on the “warped compactifications” of Calabi–Yau 5-folds by Haupt, Lukas and Stelle in [13] (see also [14]).

This paper is organised as follows. In Section 3 and after some preliminaries on Sasakian geometry in Section 2, we define the notion of a generalised Killing spinor and prove Theorem 3.2, which characterises those Sasakian spin manifolds admitting generalised Killing spinors. They turn out to be a special class of η -Einstein Sasakian manifolds with Ricci-flat transverse geometry. In Section 4 we apply this result to M-theory. We show that the class of manifolds in Section 3 almost (but not quite) provide supersymmetric backgrounds of eleven-dimensional supergravity, but subject to an additional condition on the transverse geometry, they do provide backgrounds satisfying the corrected M-theory equations and admitting generalised Killing spinors. This is the content of Theorem 4.5. We end the paper with some comments on the existence of the relevant Lorentzian Sasakian spin manifolds.

Notation. Throughout the paper, we consider Clifford algebras as defined, for instance, in [15]. According to this, the Clifford product of vectors of the standard basis of $\mathbb{R}^{p,q}$ or $\mathbb{C}^{p,q}$ is $e_i \cdot e_j + e_j \cdot e_i = -2g_{ij}$ and not “ $+2g_{ij}$ ”.

2. PRELIMINARIES ON SASAKIAN MANIFOLDS

Let M be a smooth manifold of dimension $2n + 1$ that is endowed with a metric g either positive-definite (we set $\epsilon = 1$ in this case) or Lorentzian (namely with signature $(2n, 1)$ and we set $\epsilon = -1$).

Definition 2.1. ([16, 17]) The pair (M, g) is called a *Sasakian manifold* if there exists a vector field ξ such that

- (i) ξ is a Killing vector field of constant length $\epsilon = g(\xi, \xi)$, and
- (ii) the endomorphism $\Phi = -\nabla \xi : TM \rightarrow TM$ satisfies the following two conditions for all vectors $X, Y \in TM$:

$$\Phi^2(X) = -X + \epsilon g(\xi, X)\xi$$

and

$$(\nabla_X \Phi)(Y) = \epsilon g(X, Y)\xi - \epsilon g(\xi, Y)X .$$

The Killing vector field ξ is called the *characteristic vector field* and it gives rise to a subbundle \mathcal{V} on M , $\mathcal{V}|_x = \mathbb{R}\xi|_x$, called the *vertical subbundle*. The collection of its 1-dimensional leaves defines a foliation \mathcal{F}_ξ on M which we always assume to be regular, i.e., each point of M has a neighbourhood where each leaf passes at most one time.

Every Sasakian manifold comes with the *characteristic 1-form* $\eta \in \Lambda^1(M)$ defined by $\eta(X) = \epsilon g(\xi, X)$ for all $X \in TM$. It is a contact 1-form and the associated maximally nonintegrable distribution $\mathcal{H} = \text{Ker } \eta$ is called the *horizontal subbundle*.

The tangent bundle of M decomposes into the g -orthogonal direct sum $TM = \mathcal{H} \oplus \mathcal{V}$ of the horizontal and vertical subbundles and $g = g|_{\mathcal{H}} + \epsilon \eta \otimes \eta$. We call the associated sections *horizontal* and, respectively, *vertical* vector fields on M . Some basic properties of Sasakian manifolds are

$$\begin{aligned} \Phi(\xi) &= 0, & \eta(\Phi(X)) &= 0, \\ g(\Phi(X), \Phi(Y)) &= g(X, Y) - \epsilon \eta(X)\eta(Y), \\ d\eta(X, Y) &= -2\epsilon g(\Phi(X), Y), \\ \iota_\xi d\eta &= 0, & \iota_\xi \eta &= 1. \end{aligned}$$

The last equation says that ξ is the Reeb vector field associated with η .

One important feature of a (regular) Sasakian manifold is the fact that the restriction $\epsilon \cdot \Phi|_{\mathcal{H}}$ of $\epsilon \cdot \Phi$ to \mathcal{H} is an integrable complex structure which naturally turns the transverse geometry of the characteristic foliation into a Kähler manifold.

Theorem 2.2. *Let (M, g, ξ, Φ, η) be a $2n+1$ dimensional compact Sasakian manifold. If B is the leaf space of \mathcal{F}_ξ with the natural projection $\pi : M \rightarrow B$ then:*

- (i) *B is a compact complex manifold with a Kähler metric h and integral Kähler form ω satisfying $d\eta = -2\pi^*\omega$, and*
- (ii) *$\pi : (M, g) \rightarrow (B, h)$ is a pseudo-Riemannian submersion with totally geodesic fibers all diffeomorphic to S^1 and it has the natural structure of a principal circle bundle.*

This result is usually stated in the Riemannian setting (see [18]). However it also holds in the Lorentzian case, as a simple consequence of the fact that Sasakian structures with $\epsilon = 1$ and $\epsilon = -1$ are in a one-to-one correspondence with each other.

Proposition 2.3. *Let (M, g, ξ, Φ, η) be a Sasakian manifold with $\epsilon = \pm 1$. Then $(M, \bar{g}, \xi, \bar{\Phi}, \eta)$ is a Sasakian manifold with $\bar{\epsilon} = -\epsilon$, where $\bar{g} = g - 2\epsilon \eta \otimes \eta$ and $\bar{\Phi} = -\Phi$. Moreover the Levi-Civita connection of g and \bar{g} are related as $\bar{\nabla}_X Y = \nabla_X Y + 2\eta(Y)\Phi(X) + 2\eta(X)\Phi(Y)$.*

There is an inverse construction to Theorem 2.2, which we briefly recall. The *contactification* of a symplectic manifold (B, ω) is a contact manifold of one dimension higher, introduced as the total space M of an appropriate

bundle $\pi : M \rightarrow B$ endowed with a contact structure $\mathcal{H} = \text{Ker } \eta$ (see, e.g., [19, 20]).

If $\omega = d\alpha$ is exact one can simply consider the trivial bundle $M = B \times \mathbb{R}$ and the 1-form $\eta = dt - 2\alpha$, where t is the coordinate on \mathbb{R} . If ω represents an integral cohomology class, Boothby and Wang first considered a principal circle bundle $\pi : M \rightarrow B$ of Euler class $[\omega]$ and then a connection 1-form A with curvature $dA = -2\pi i \pi^* \omega$; the 1-form $\eta = -\frac{i}{\pi} A$ is the required contact form on M . For more details on this construction we refer to [21].

If B is in addition Kähler with respect to a complex structure J then $DJ = D\omega = 0$ where D is the Levi-Civita connection of the corresponding Hermitian metric $h(\cdot, \cdot) = \omega(\cdot, J\cdot)$ and the contactification (M, η) has a natural Sasakian structure with horizontal subbundle $\mathcal{H} = \text{Ker } \eta$ and the following tensor fields:

- (i) $g = \pi^* h + \epsilon \eta \otimes \eta$,
- (ii) ξ is the fundamental field of the action of S^1 on M (of period 2),
and
- (iii) Φ is the $(1, 1)$ -tensor uniquely determined by $\Phi|_{\mathcal{H}} = \epsilon \pi^* J$, $\Phi(\xi) = 0$.

Any Sasakian structure obtained in this way is called *strongly regular*.

We conclude this section with a basic result on the curvature tensors of a strongly regular Sasakian manifold.

We first fix some notation: we denote by $\widehat{U} \in \mathfrak{X}(M)$ the basic lift of a vector field $U \in \mathfrak{X}(B)$ on the base of $\pi : M \rightarrow B$, this is the unique horizontal vector field satisfying

$$\pi_* \widehat{U} = U \quad \text{and} \quad \mathcal{L}_\xi \widehat{U} = 0. \quad (2.1)$$

The bracket of two basic lifts is

$$[\widehat{U}, \widehat{V}] = \widehat{[U, V]} + 2\omega(U, V)\xi. \quad (2.2)$$

Proposition 2.4. *Let (M, g, ξ, Φ, η) be a strongly regular Sasakian manifold with $\epsilon = \pm 1$ and Kähler transverse geometry (B, h, J, ω) . Then*

- (i) *the Levi-Civita connection ∇ of g is the unique linear connection which satisfies*

$$\begin{aligned} \nabla_\xi \xi &= 0 \\ \nabla_{\widehat{U}} \xi &= \nabla_\xi \widehat{U} = -\epsilon J \widehat{U} \\ \nabla_{\widehat{U}} \widehat{V} &= \widehat{D_U V} + \omega(U, V)\xi; \end{aligned} \quad (2.3)$$

(ii) the curvature tensor R of g is the unique $(1,3)$ -tensor field which satisfies

$$\begin{aligned} R(\xi, \widehat{V})\xi &= -\widehat{V}\xi \\ R(\xi, \widehat{V})\widehat{W} &= \epsilon h(V, W)\xi \\ R(\widehat{U}, \widehat{V})\xi &= 0 \\ R(\widehat{U}, \widehat{V})\widehat{W} &= (R_h(U, V)W)^\wedge + 2\epsilon\omega(U, V)\widehat{JW} \\ &\quad - \epsilon(\omega(V, W)\widehat{JU} - \omega(U, W)\widehat{JV}) , \end{aligned} \tag{2.4}$$

where R_h is the curvature tensor of h ;

(iii) the Ricci curvature Ric of g is the unique symmetric $(0,2)$ -tensor field which satisfies

$$\begin{aligned} \text{Ric}(\xi, \xi) &= 2n \\ \text{Ric}(\xi, \widehat{U}) &= 0 \\ \text{Ric}(\widehat{U}, \widehat{U}) &= \text{Ric}_h(U, U) - 2\epsilon h(U, U) , \end{aligned} \tag{2.5}$$

where Ric_h is the Ricci curvature of h .

Points (i) and (ii) are proved by direct computations which use (2.1), (2.2) and the fact that (B, h, ω, J) is Kähler. To prove (iii) it is convenient to consider local *adapted frames*, namely oriented g -orthonormal frames on M of the form (\widehat{e}_i, ξ) for some h -orthonormal frame $(e_i)_{i=1}^{2n}$ on B with $e_{i+n} = Je_i$ for all $1 \leq i \leq n$. We omit the details for the sake of brevity.

From now on we will tacitly restrict to strongly regular Sasakian manifolds, but recall that every regular and compact Sasakian manifold is automatically strongly regular by Theorem 2.2.

3. NULL SASAKIAN GEOMETRY AND GENERALISED KILLING SPINORS

The main aim of this section is to prove Theorem 3.2, a spinorial characterisation of Sasakian structures with Ricci flat transverse geometry. We remark that by Proposition 2.4 such structures are never Einstein but rather η -Einstein (see, e.g., [22]) since they satisfy $\text{Ric} = \lambda g + \nu \eta \otimes \eta$ with $\lambda = -2\epsilon$ and $\nu = 2(n+1)$. In particular, Bär's cone construction [8], relating the existence of real Killing spinors to special holonomy cones, does not apply in our case. Instead, the relevant definition is the following.

Definition 3.1. Let (M, g, ξ, Φ, η) be a $2n+1$ -dimensional Sasakian spin manifold with $\epsilon = \pm 1$. A *generalised Killing spinor* is a non-zero section φ of the associated Dirac spinor bundle $p: \mathbb{S}(M) \rightarrow M$ which satisfies:

- (a) $\nabla_X \varphi = \frac{1}{2}\epsilon \Phi(X) \cdot \xi \cdot \varphi$ for all horizontal vectors X ;
- (b) $\nabla_\xi \varphi = -\Phi \cdot \varphi$ where Φ is understood as an element of $\mathfrak{so}(TM)$.

Note that any generalised Killing spinor is nowhere vanishing since it is non-zero by definition and parallel with respect to a connection on the spinor bundle. Our main result is the following

Theorem 3.2. *Let (M, g, ξ, Φ, η) be a $2n+1$ -dimensional Sasakian manifold with $\epsilon = \pm 1$ and transverse Kähler geometry (B, h, J, ω) . Then M admits a spin structure if and only if B admits a spin structure and*

- (i) *if M has a spin structure carrying a generalised Killing spinor then $\text{Ric}_h = 0$;*
- (ii) *conversely if B has a spin structure carrying a parallel spinor then $\text{Ric}_h = 0$ and M has a generalised Killing spinor.*

Moreover if B has full holonomy $\text{SU}(n)$ or it is the standard complex torus then there are two parallel spinors such that the corresponding generalised Killing spinors φ_{\pm} on M satisfy $\Phi \cdot \varphi_{\pm} = \pm \epsilon i \frac{n}{2} \varphi_{\pm}$.

The remaining part of this section is essentially devoted to the proof of Theorem 3.2. We first show that the existence of a generalised Killing spinor forces the transverse geometry to be Ricci-flat and then prove (ii) and the last claim.

Let M be a Sasakian spin manifold. Recall that the “T-trace” $\text{Tr}_{\Gamma}(R)$ of the curvature tensor is the 1-form on M with values in the endomorphisms bundle of $\mathbb{S}(M)$ defined by

$$\text{Tr}_{\Gamma}(R)(X) \cdot \varphi := (\epsilon \xi \cdot R(X, \xi) + \sum_{i=1}^{2n} \widehat{e}_i \cdot R(X, \widehat{e}_i)) \cdot \varphi ,$$

for all $X \in TM$ and $\varphi \in \mathbb{S}(M)$, where (\widehat{e}_i, ξ) is an adapted frame on M . If r is the $(1, 1)$ -Ricci curvature tensor of g , standard arguments and part (iii) of Proposition 2.4 yield the general identity

$$\text{Tr}_{\Gamma}(R)(X) \cdot \varphi = -\frac{1}{2}r(X) \cdot \varphi = \begin{cases} -n\epsilon \xi \cdot \varphi & \text{if } X = \xi ; \\ -\frac{1}{2}\widehat{r_h(U)} \cdot \varphi + \epsilon \widehat{U} \cdot \varphi & \text{if } X = \widehat{U} , \end{cases} \quad (3.1)$$

where r_h is the $(1, 1)$ -Ricci curvature tensor of h .

The expression of the curvature tensor in part (ii) of Proposition 2.4 yields also

$$R(\widehat{U}, \xi) \cdot \varphi = \frac{1}{2}\epsilon \xi \cdot \widehat{U} \cdot \varphi .$$

Now, if φ is generalised Killing

$$\begin{aligned} \nabla_{\widehat{U}} \nabla_{\widehat{V}} \varphi &= \nabla_{\widehat{U}} (\frac{1}{2} \widehat{JV} \cdot \xi \cdot \varphi) \\ &= \frac{1}{2} \nabla_{\widehat{U}} \widehat{JV} \cdot \xi \cdot \varphi + \frac{1}{2} \widehat{JV} \cdot \nabla_{\widehat{U}} \xi \cdot \varphi + \frac{1}{2} \widehat{JV} \cdot \xi \cdot \nabla_{\widehat{U}} \varphi \\ &= \frac{1}{2} \nabla_{\widehat{U}} \widehat{JV} \cdot \xi \cdot \varphi + \frac{1}{2} \widehat{JV} \cdot \nabla_{\widehat{U}} \xi \cdot \varphi + \frac{1}{4} \epsilon \widehat{JV} \cdot \widehat{JU} \cdot \varphi \\ &= \frac{1}{2} \widehat{D_U JV} \cdot \xi \cdot \varphi - \frac{1}{2} \epsilon h(U, V) \varphi - \frac{1}{4} \epsilon \widehat{JV} \cdot \widehat{JU} \cdot \varphi \\ &= \frac{1}{2} \widehat{JD_U V} \cdot \xi \cdot \varphi - \frac{1}{2} \epsilon h(U, V) \varphi - \frac{1}{4} \epsilon \widehat{JV} \cdot \widehat{JU} \cdot \varphi \end{aligned}$$

and, using (2.2) and property (b) of Definition 3.1,

$$\begin{aligned} R(\widehat{U}, \widehat{V}) \cdot \varphi &= 2\omega(U, V) \Phi \cdot \varphi - \frac{1}{4} \epsilon (\widehat{JV} \cdot \widehat{JU} - \widehat{JU} \cdot \widehat{JV}) \cdot \varphi \\ &= 2h(JU, V) \Phi \cdot \varphi - \frac{1}{2} \epsilon (h(U, V) + \widehat{JV} \cdot \widehat{JU}) \cdot \varphi . \end{aligned}$$

The value of the Γ -trace on a generalised Killing spinor φ and an horizontal lift \widehat{U} is therefore equal to

$$\begin{aligned}
\mathrm{Tr}_\Gamma(R)(\widehat{U}) \cdot \varphi &= (\epsilon \xi \cdot R(\widehat{U}, \xi) + \sum_{i=1}^{2n} \widehat{e}_i \cdot R(\widehat{U}, \widehat{e}_i)) \cdot \varphi \\
&= -\frac{1}{2} \epsilon \widehat{U} \cdot \varphi + 2 \sum_{i=1}^{2n} \widehat{e}_i \cdot h(JU, e_i) \Phi \cdot \varphi \\
&\quad - \frac{1}{2} \sum_{i=1}^{2n} \widehat{e}_i \cdot \epsilon(h(U, e_i) + \widehat{J}e_i \cdot \widehat{JU}) \cdot \varphi \\
&= -\frac{1}{2} \epsilon \widehat{U} \cdot \varphi + 2 \widehat{JU} \cdot \Phi \cdot \varphi - \frac{1}{2} \epsilon \widehat{U} \cdot \varphi - \sum_{i=1}^n \epsilon \widehat{e}_i \cdot \widehat{J}e_i \cdot \widehat{JU} \cdot \varphi \\
&= -\epsilon \widehat{U} \cdot \varphi + 2 \widehat{JU} \cdot \Phi \cdot \varphi - 2 \Phi \cdot \widehat{JU} \cdot \varphi \\
&= -\epsilon \widehat{U} \cdot \varphi + 2 \widehat{JU} \cdot \Phi \cdot \varphi - 2 \widehat{JU} \cdot \Phi \cdot \varphi + 2 \epsilon \widehat{U} \cdot \varphi \\
&= \epsilon \widehat{U} \cdot \varphi .
\end{aligned}$$

Comparing this with (3.1) immediately yields

$$\widehat{r_h(U)} \cdot \varphi = 0$$

for all $U \in \mathfrak{X}(B)$ and $\widehat{r_h(U)}$ is a null vector of the distribution \mathcal{H} . As $g|_{\mathcal{H}}$ is positive-definite one gets $\widehat{r_h(U)} = 0$ and $\mathrm{Ric}_h = 0$.

We now describe the relation between the spin structure of the total space and the base of the characteristic fibration $\pi : (M, g) \rightarrow (B, h)$. We recall here only the facts that we need and refer to e.g. [23, §5] for more details on spin structures and pseudo-Riemannian submersions.

The orthogonal splitting $TM = \mathcal{H} \oplus \mathcal{V}$ of the tangent bundle of M and the fact that $\mathcal{V} = \mathrm{Ker} \pi_*$ is trivialised by ξ define an $\mathrm{SO}(2n)$ -reduction $P_\sharp \subset P_g$ of the bundle $p : P_g \rightarrow M$ of oriented g -orthonormal frames on M , where

$$\mathcal{H} = P_\sharp \times_{\mathrm{SO}(2n)} \mathbb{R}^{2n} \quad \text{and} \quad P_\sharp / \mathrm{SO}(2n)|_x \simeq \xi|_x .$$

If P_h is the bundle of oriented h -orthonormal frames on B , the natural map

$$d\pi : P_\sharp \rightarrow P_h , \quad d\pi(u) = \pi_* \circ u|_{\mathbb{R}^{2n}} ,$$

is an isomorphism on each fiber and it identifies $P_\sharp \simeq \pi^* P_h$.

We say that a (local) section $\widehat{s} : M \rightarrow P_\sharp$ is *basic* if there exists a section $s : B \rightarrow P_h$ with $d\pi \circ \widehat{s} = s \circ \pi$; the basic sections and the sections of P_h are in a one-to-one correspondence. Similarly, if W is an $\mathrm{SO}(2n)$ -module, an equivariant map $\widehat{f} : P_\sharp \rightarrow W$ is *basic* if there exists $f : P_h \rightarrow W$ such that $\widehat{f} = f \circ d\pi$. In this case the sections

$$\begin{aligned}
\widehat{\varphi} &= [\widehat{s}, \widehat{f} \circ \widehat{s}] : M \rightarrow P_\sharp \times_{\mathrm{SO}(2n)} W & \text{and} \\
\varphi &= [s, f \circ s] : B \rightarrow P_h \times_{\mathrm{SO}(2n)} W
\end{aligned} \tag{3.2}$$

of the associated bundles $P_{\sharp} \times_{\mathrm{SO}(2n)} W \rightarrow M$ and $P_h \times_{\mathrm{SO}(2n)} W \rightarrow B$, respectively, are related by the identity $d\pi \circ \widehat{\varphi} = \varphi \circ \pi$ and all sections $\widehat{\varphi}$ of $P_{\sharp} \times_{\mathrm{SO}(2n)} W$ which are basic (i.e., they satisfy this identity for some φ) are as in (3.2).

We also say that two elements $u \in P_{\sharp}|_x$ and $u' \in P_{\sharp}|_{x'}$ are *equivalent*, written $u \sim u'$, if $\pi(x) = \pi(x')$ and there is a local basic section \widehat{s} with $\widehat{s}(x) = u$ and $\widehat{s}(x') = u'$. The bundle P_h and its sections are then naturally identifiable with P_{\sharp}/\sim and, respectively, the basic sections of P_{\sharp} .

Let now $\mathrm{Ad} : \widetilde{P}_g \rightarrow P_g$ be a spin structure on M , a double cover of P_g with structure group \widetilde{G} isomorphic to $\mathrm{Spin}(2n+1)$ if $\epsilon = 1$ or to $\mathrm{Spin}(2n,1)$ if $\epsilon = -1$, inducing the canonical covering morphism on each fiber. We consider also the principal bundle $\widetilde{P}_{\sharp} := \mathrm{Ad}^{-1}(P_{\sharp})$ on M with fiber $\mathrm{Spin}(2n) = \mathrm{Ad}^{-1}(\mathrm{SO}(2n))$. In complete analogy with the bundles of linear frames, two elements $u \in \widetilde{P}_{\sharp}|_x$ and $u' \in \widetilde{P}_{\sharp}|_{x'}$ are called equivalent if $\pi(x) = \pi(x')$ and there is a local section $\widehat{s} : M \rightarrow \widetilde{P}_{\sharp}$ with $\widehat{s}(x) = u$, $\widehat{s}(x') = u'$ and which is basic, in the sense that $\mathrm{Ad}(\widehat{s}) : M \rightarrow P_{\sharp}$ is basic. By [23, Lemma 5]

$$\mathrm{Ad} : \widetilde{P}_h := \widetilde{P}_{\sharp}/\sim \longrightarrow P_h \simeq P_{\sharp}/\sim$$

is a two-sheeted covering and therefore a spin structure on B (the proof of the lemma is given just in the Riemannian case but it extends verbatim to the Lorentzian case too).

Conversely if B is endowed with a spin structure $\mathrm{Ad} : \widetilde{P}_h \rightarrow P_h$, the pull-back bundle $\widetilde{P}_{\sharp} := \pi^* \widetilde{P}_h$ is a double cover of P_{\sharp} and enlarging its structure group $\mathrm{Spin}(2n)$ to \widetilde{G} yields the spin structure $\widetilde{P}_g = \widetilde{P}_{\sharp} \times_{\mathrm{Spin}(2n)} \widetilde{G}$ on M . This argument shows that M admits a spin structure *if and only if* B does.

Let \mathbb{S} be an irreducible module for the complex Clifford algebra $\mathbb{C}l(2n+1)$ so that the Dirac spinor fields on M are given by the sections of the bundle $\mathbb{S}(M) = \widetilde{P}_g \times_{\widetilde{G}} \mathbb{S} \simeq \widetilde{P}_{\sharp} \times_{\mathrm{Spin}(2n)} \mathbb{S}$. As \mathbb{S} is irreducible also for $\mathbb{C}l(2n)$ (see the classification of Clifford algebras in e.g. [15]), the *basic sections* $\widehat{\varphi} : M \rightarrow \widetilde{P}_{\sharp} \times_{\mathrm{Spin}(2n)} \mathbb{S}$ of the very same bundle are in a one-to-one correspondence with the spinor fields on B and form a natural subclass of the spinors on M . Similarly a section of the bundle of Clifford algebras on B is represented by a basic section of $\widetilde{P}_{\sharp} \times_{\mathrm{Spin}(2n)} \mathbb{C}l(2n+1) \rightarrow M$ with values in $\mathbb{C}l(2n)$.

Our aim is to determine the covariant derivatives $D\varphi$ and $\nabla\widehat{\varphi}$ of a spinor φ on B and the corresponding (basic) spinor $\widehat{\varphi}$ on M . Let $\widehat{\vartheta} : TP_g \rightarrow \mathbb{R}^{2n+1}$ be the so-called soldering form of P_g , defined by $\widehat{\vartheta}_u(v) = (v^1, \dots, v^{2n+1})$ where the v^i are the components of $p_*(v) \in T_{p(u)}M$ with respect to the frame u . The restriction of the soldering form to P_{\sharp} decomposes into

$$\widehat{\vartheta}|_{P_{\sharp}} = (\vartheta_{\mathcal{V}}, \vartheta_{\mathcal{H}}), \quad \vartheta_{\mathcal{H}} = (d\pi)^* \vartheta,$$

where $\vartheta : TP_h \rightarrow \mathbb{R}^{2n}$ is the soldering form of P_h . We call a vector $v \in T_u P_{\sharp}$ *horizontal* (resp. *vertical*) if $\vartheta_{\mathcal{V}}(v) = 0$ (resp. $\vartheta_{\mathcal{H}}(v) = 0$).

Let also $\widehat{\omega}$ and ω be the Levi-Civita connection 1-forms on P_g and P_h , respectively. One has the decomposition

$$\widehat{\omega}|_{P_{\sharp}} = \begin{pmatrix} \omega_{\mathcal{H}} & A \\ -\epsilon A^T & 0 \end{pmatrix} \quad \text{where} \quad A : TP_{\sharp} \rightarrow \mathbb{R}^{2n} \quad \text{and} \quad \omega_{\mathcal{H}} : TP_{\sharp} \rightarrow \mathfrak{so}(2n),$$

and $\omega_{\mathcal{H}}(v) = (d\pi)^*\omega(v)$ for any horizontal $v \in TP_{\sharp}$ (see [23, Lemma 4]).

Proposition 3.3. *Let φ be a spinor on B and $\widehat{\varphi}$ the corresponding (basic) spinor on M . Then we have*

- (i) $\nabla_{\widehat{U}}\widehat{\varphi} = (D_U\varphi)^{\widehat{\cdot}} + \frac{1}{2}\epsilon\Phi(\widehat{U}) \cdot \xi \cdot \widehat{\varphi}$ for all $U \in \mathfrak{X}(B)$;
- (ii) $\nabla_{\xi}\widehat{\varphi} = -\Phi \cdot \widehat{\varphi}$.

If φ is parallel then $\text{Ric}_h = 0$ and $\widehat{\varphi}$ is a generalised Killing spinor.

Proof. By definitions $\widehat{\varphi} = [\widehat{s}, \pi^*\Psi]$ for some basic section $\widehat{s} : M \rightarrow \widetilde{P}_{\sharp}$ and a map $\Psi : B \rightarrow \mathbb{S}$. By the second part of [23, Lemma 5] and [23, Prop. 3] one has for all basic lifts \widehat{U} on M

$$\begin{aligned} \nabla_{\widehat{U}}\widehat{\varphi} &= \nabla_{\widehat{U}}[\widehat{s}, \pi^*\Psi] = [\widehat{s}, (\widehat{s}^*\widehat{\omega})(\widehat{U}) \cdot \pi^*\Psi + \pi^*d\Psi(\widehat{U})] \\ &= (D_U\varphi)^{\widehat{\cdot}} - \frac{1}{2}\epsilon\widehat{s}^*A(\widehat{U}) \cdot \xi \cdot [\widehat{s}, \pi^*\Psi] \\ &= (D_U\varphi)^{\widehat{\cdot}} + \frac{1}{2}\epsilon\Phi(\widehat{U}) \cdot \xi \cdot \widehat{\varphi}. \end{aligned}$$

A similar computation for the Reeb vector field yields

$$\begin{aligned} \nabla_{\xi}\widehat{\varphi} &= \nabla_{\xi}[\widehat{s}, \pi^*\Psi] = [\widehat{s}, (\widehat{s}^*\widehat{\omega})(\xi) \cdot \pi^*\Psi + \pi^*d\Psi(\xi)] \\ &= [\widehat{s}, (\widehat{s}^*\widehat{\omega})(\xi) \cdot \pi^*\Psi] \stackrel{(\nabla_{\xi}\xi=0)}{=} [\widehat{s}, (\widehat{s}^*\omega_{\mathcal{H}})(\xi) \cdot \pi^*\Psi] \\ &= -\Phi \cdot \widehat{\varphi}. \end{aligned}$$

This proves (i) and (ii). The last two claim follow from a standard “ Γ -trace” computation and these identities. The proof is completed. \square

We showed (i) and (ii) of Theorem 3.2. To prove the last claim we first need to recall the explicit description of the Dirac spin module. Let $W = \mathbb{R}^{2n}$ be the standard Euclidean space with orthonormal basis $(e_i)_{i=1}^{2n}$ and set $V = W \oplus \mathbb{R}\xi$ with $g(\xi, \xi) = \epsilon$. The complexification $W^{\mathbb{C}} = W \otimes \mathbb{C}$ of W decomposes as a direct sum of isotropic subspaces $W^{\mathbb{C}} = W^{10} \oplus W^{01}$, where

$$W^{10} = \langle e_i^{10} = \frac{1}{2}(e_i - ie_{i+n}) \mid i = 1, \dots, n \rangle \quad \text{and} \quad W^{01} = \overline{W^{10}}.$$

Let $U = W^{10}$ and set $\mathbb{S} = \Lambda^{\bullet}U^*$. For any $v, v' \in U$, $w, w' \in \overline{U}$ and $\varphi \in \mathbb{S}$, the identities

$$v \cdot \varphi := -2\iota_v\varphi, \tag{3.3}$$

$$w \cdot \varphi := w^{\flat} \wedge \varphi, \tag{3.4}$$

satisfy $v v' + v' v = w w' + w' w = 0$ and $v w + w v = -2g(v, w)$ and therefore give an irreducible representation of the Clifford algebra $\text{Cl}(2n) \simeq \mathbb{C}(2^n)$. This representation splits in the direct sum $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ of two irreducible Weyl spinor modules $\mathbb{S}^+ = \Lambda^{\text{even}}U^*$ and $\mathbb{S}^- = \Lambda^{\text{odd}}U^*$ for the even part of

$\mathbb{C}l(2n)$; they can also be intrinsically described as the ± 1 -eigenspaces of the volume element $\text{vol}_{2n} := (-1)^{\frac{n(n+1)}{2}} i^n e_1 \cdots e_{2n}$.

To obtain an irreducible representation of $\mathbb{C}l(2n+1) \simeq \mathbb{C}(2^n) \oplus \mathbb{C}(2^n)$, it is sufficient to complement (3.3) and (3.4) with

$$\xi \cdot \varphi := i^{\frac{\epsilon+1}{2}} \text{vol}_{2n} \cdot \varphi . \quad (3.5)$$

Now, if one fixes an adapted local frame at $x \in M$, the action of $\Phi \in \mathfrak{so}(V)$ on a spinor $\varphi \in \Lambda^p U^*$ is given by

$$\begin{aligned} \frac{1}{2} \epsilon \Phi \cdot \varphi &= -i \sum_{i=1}^n (e_i^{10} \wedge e_i^{01}) \cdot \varphi = -\frac{i}{4} \sum_{i=1}^n [e_i^{10}, e_i^{01}] \cdot \varphi \\ &= \frac{i}{2} \sum_{i=1}^n \iota_{e_i^{10}} ((e_i^{01})^\flat \wedge \varphi) - \frac{i}{2} \sum_{i=1}^n (e_i^{01})^\flat \wedge \iota_{e_i^{10}} \varphi \\ &= \frac{n}{4} i \varphi - i \sum_{i=1}^n (e_i^{01})^\flat \wedge \iota_{e_i^{10}} \varphi = \frac{n}{4} i \varphi - \frac{p}{2} i \varphi \\ &= \frac{n-2p}{4} i \varphi . \end{aligned} \quad (3.6)$$

If B is simply-connected with holonomy $\text{SU}(n)$ then it is spin and it is endowed with two parallel spinors φ_\pm such that $\varphi_+ \in \Lambda^0 U^*$ and $\varphi_- \in \Lambda^n U^*$ at any point (see [24, pag. 61]). In the non simply-connected case or if B is the standard complex torus the same is true, provided an appropriate spin structure is chosen (see [25, 26] for the first case and consider the products of the periodic ‘‘Ramond’’ spin structures of the circle S^1 in the second case).

These observations and (3.6) imply at once the last part of Theorem 3.2. We collect here, for later use in Section 4, the equations that are satisfied by the two generalised Killing spinors φ_\pm :

$$\nabla_X \varphi_\pm = \frac{1}{2} \epsilon \Phi(X) \cdot \xi \cdot \varphi_\pm \quad (X \in \mathcal{H}) , \quad \nabla_\xi \varphi_\pm = \mp \epsilon i \frac{n}{2} \varphi_\pm . \quad (3.7)$$

4. APPLICATIONS TO SUPERSYMMETRIC THEORIES OF GRAVITY

In Section 3 we showed that the integrability conditions for the existence of a generalized Killing spinor as in Definition 3.1 correspond to the transverse geometry of the Sasakian manifold being Ricci-flat. In this section we will see that these conditions are tightly related to a generalized Einstein equation on Sasakian manifolds and, in particular, we will investigate the existence of M-theory backgrounds on Lorentzian Sasakian manifolds which are (possibly nontrivial) bundles over Calabi–Yau 5-folds.

4.1. A generalised Einstein equation on Sasakian manifolds. We recall that a bosonic background of eleven-dimensional supergravity is given by an eleven-dimensional Lorentzian manifold (M, g) endowed with a closed four form $F \in \Lambda^4 M$ subject to two partial differential equations (see [10]): the *Einstein equation*

$$\text{Ric}(X, Y) = \frac{1}{2} g(\iota_X F, \iota_Y F) - \frac{1}{6} g(X, Y) g(F, F) , \quad (4.1)$$

and the *Maxwell equation*

$$d \star F = -\frac{1}{2} F \wedge F . \quad (4.2)$$

One is usually interested in *supersymmetric* backgrounds, that is, backgrounds admitting a spin structure and a non-zero spinor $\varphi \in \Gamma(\mathbb{S}(M))$ which is pseudo-Majorana and satisfies¹

$$\nabla_X \varphi + i(\frac{1}{6} \iota_X F + \frac{1}{12} X^\flat \wedge F) \cdot \varphi = 0 ,$$

for all $X \in TM$. We recall here that φ is *pseudo-Majorana* if it satisfies the reality condition $j(\varphi) = \varphi$ where $j : \mathbb{S}(M) \rightarrow \mathbb{S}(M)$ is an appropriate antilinear involution on the space of Dirac spinors (see, e.g., [27]). This map can be conveniently described fixing an adapted local frame at $x \in M$ and identifying each fiber $\mathbb{S}(M)|_x$ with our model (3.3)–(3.5) of the Dirac spin module $\mathbb{S} = \Lambda^\bullet U^*$. Consider first the “Hodge star operator” $\star : \mathbb{S} \rightarrow \mathbb{S}$ given by

$$\star(\Lambda^p U^*) \subset \Lambda^{5-p} U^* \quad \text{and} \quad \varphi \wedge \star \varphi' = g(\varphi, \overline{\varphi}') \varphi_- ,$$

where $\varphi \rightarrow \overline{\varphi} \in \Lambda^\bullet \overline{U}^*$ is the standard conjugation, $\varphi, \varphi' \in \mathbb{S}$ and $\varphi_- \in \Lambda^5 U^*$ is normalised so that $g(\varphi_-, \overline{\varphi}_-) = 1$. One can check that \star is an antilinear involution which is not $\text{Spin}(10, 1)$ -equivariant as it satisfies

$$\begin{aligned} \star(v \cdot \varphi) &= 2(-1)^{|\varphi|} \overline{v} \cdot \star \varphi , \\ \star(w \cdot \varphi) &= \frac{1}{2}(-1)^{|\varphi|+1} \overline{w} \cdot \star \varphi , \\ \star(\xi \cdot \varphi) &= -\xi \cdot \star \varphi , \end{aligned}$$

for all $v \in U$ and $w \in \overline{U}$. However, if one sets

$$j|_{\Lambda^p U^*} := \frac{(-1)^{\frac{p(p-1)}{2}} 2^p}{\sqrt{32}} \star|_{\Lambda^p U^*} ,$$

then $j : \mathbb{S} \rightarrow \mathbb{S}$ is also antilinear, $j^2 = \text{Id}$ and

$$\begin{aligned} j(v \cdot \varphi) &= -\overline{v} \cdot j \varphi , \\ j(w \cdot \varphi) &= -\overline{w} \cdot j \varphi , \\ j(\xi \cdot \varphi) &= -\xi \cdot j \varphi . \end{aligned}$$

It follows that this map is $\text{Spin}(10, 1)$ -equivariant and it induces the required pseudo-Majorana conjugation $j : \mathbb{S}(M) \rightarrow \mathbb{S}(M)$. We remark for later use that $j(\Lambda^0 U^*) \subset \Lambda^5 U^*$ at any point $x \in M$.

The equations (4.2) receive higher order corrections in M-theory. Before turning to them, we first focus on (4.1), a modification of the classical Einstein equations which makes sense on any Sasakian manifold of dimension $2n + 1$, $\epsilon = \pm 1$. As usual we denote the associated characteristic fibration by

$$\pi : (M, g, \xi, \Phi, \eta) \rightarrow (B, h, J, \omega) ,$$

¹The perhaps unusual form of this equation is due to our conventions on Clifford algebras and our metric conventions being “mostly plus”.

and our ansatz on the flux is

$$F = \lambda \pi^* \omega^2, \quad (4.3)$$

where λ is some real constant. This form is exact, and hence closed, by (i) of Theorem 2.2.

Theorem 4.1. *A Sasakian manifold is a solution of the Einstein equations with $F = \lambda \pi^* \omega^2$ if and only if $\lambda^2 = -\frac{6\epsilon}{n-1}$ and $\text{Ric}_h = 2\epsilon(n-5)h$. If this is the case the metric g is necessarily Lorentzian and the flux is non-zero.*

Proof. According to the orthogonal decomposition $TM = \mathcal{H} \oplus \mathcal{V}$, the equation (4.1) splits into three components. The one with $X = \xi$ and Y horizontal is automatically satisfied, since $\text{Ric}(\xi, \mathcal{H}) = 0$ (see (2.5)) and $\iota_\xi F = 0$.

Fix now an h -orthonormal frame $(e_i)_{i=1}^{2n}$ on B which satisfies $e_{i+n} = J e_i$ for all $1 \leq i \leq n$ and let $(e^i)_{i=1}^{2n}$ be the corresponding h -dual frame so that

$$\omega = \sum_{i=1}^n e^i \wedge e^{i+n}.$$

One has

$$\begin{aligned} g(F, F) &= \lambda^2 h(\omega^2, \omega^2) \\ &= \lambda^2 h\left(\sum_{i,j=1}^n e^i \wedge e^{i+n} \wedge e^j \wedge e^{j+n}, \sum_{i,j=1}^n e^i \wedge e^{i+n} \wedge e^j \wedge e^{j+n}\right) \\ &= 2n(n-1)\lambda^2 \end{aligned}$$

and, by a similar computation, $g(\iota_{\widehat{U}} F, \iota_{\widehat{U}} F) = 4(n-1)\lambda^2 h(U, U)$ for any horizontal lift \widehat{U} . This and the last equation of (2.5) yield that the \mathcal{H} -component of the Einstein equations is satisfied if and only if

$$\text{Ric}_h = \left(\frac{1}{3}(6-n)(n-1)\lambda^2 + 2\epsilon\right) h. \quad (4.4)$$

Finally the \mathcal{V} -component holds if and only if

$$\lambda^2 = -\frac{6\epsilon}{n-1} \quad (4.5)$$

as $\text{Ric}(\xi, \xi) = 2n$ and $\frac{1}{2}g(\iota_\xi F, \iota_\xi F) - \frac{1}{6}g(\xi, \xi)g(F, F) = -\frac{1}{3}\epsilon n(n-1)\lambda^2$. This gives the last two claims of the theorem and by substituting (4.5) in (4.4) also $\text{Ric}_h = 2\epsilon(n-5)h$. \square

We note that the base of the characteristic fibration of a solution of the Einstein equations is Kähler-Einstein and it is Ricci-flat precisely in the 11-dimensional case. It is this fact that will ultimately allow us to use Theorem 3.2 in the case $n = 5$, $\epsilon = -1$ and to discuss the existence of a particular kind of pseudo-Majorana spinors; before doing so we have a look to the equations (4.2).

From now on and in the rest of the paper we restrict to the 11-dimensional Lorentzian case.

First of all

$$-\frac{1}{2}F \wedge F = -\frac{1}{2}\lambda^2 \pi^* \omega^4 .$$

To compute the l.h.s. of (4.2), we need two relations:

- (a) the volume of g is $\text{vol} = -\eta \wedge \pi^* \text{vol}_h$, where vol_h is the volume of h , and
- (b) for any p -form β on B one has $\star(\pi^* \beta) = -(-1)^p \eta \wedge \pi^*(\star \beta)$.

To see these fix an adapted frame (\widehat{e}_i, ξ) and the associated g -dual frame $(\widehat{e}^i, -\eta)$. Then (a) is a consequence of the identities $\text{vol}_h = e^1 \wedge \dots \wedge e^{2n}$, $\text{vol} = -\eta \wedge \widehat{e}^1 \wedge \dots \wedge \widehat{e}^{10}$ and $\widehat{e}^i = \pi^* e^i$. Now $\widehat{\alpha} \wedge \star(\pi^* \beta) = g(\widehat{\alpha}, \pi^* \beta) \text{vol}$ for any p -form $\widehat{\alpha}$ on M and

$$g(\widehat{\alpha}, \pi^* \beta) = \begin{cases} 0 & \text{if } \widehat{\alpha} = \eta \wedge \pi^* \alpha \text{ for some } \alpha \in \Lambda^{p-1} B ; \\ h(\alpha, \beta) & \text{if } \widehat{\alpha} = \pi^* \alpha \text{ for some } \alpha \in \Lambda^p B . \end{cases}$$

Point (b) follows then from the case $\widehat{\alpha} = \pi^* \alpha$ and

$$\begin{aligned} \widehat{\alpha} \wedge \star(\pi^* \beta) &= h(\alpha, \beta) \text{vol} = -\eta \wedge h(\alpha, \beta) \pi^* \text{vol}_h \\ &= -\eta \wedge \widehat{\alpha} \wedge \pi^*(\star \beta) = \widehat{\alpha} \wedge (-(-1)^p \eta \wedge \pi^*(\star \beta)) . \end{aligned}$$

Using these two relations, one gets

$$\begin{aligned} d \star F &= \lambda d \star (\pi^* \omega^2) = -\lambda d(\eta \wedge \pi^*(\star \omega^2)) = -\frac{1}{3} \lambda d(\eta \wedge \pi^* \omega^3) \\ &= -\frac{1}{3} \lambda d\eta \wedge \pi^* \omega^3 \\ &= \frac{2}{3} \lambda \pi^* \omega^4 . \end{aligned}$$

On the other hand $\lambda^2 = \frac{3}{2}$ if the Einstein equations hold, implying that the Maxwell equations are not satisfied, at least with the right coefficients. It seems that this issue cannot be fixed in any straightforward way; we will however shortly see that one has interesting consequences on M-theory version of the supergravity equations.

4.2. M-theory on Calabi-Yau 5-folds. The first and second *Pontryagin forms* of (M, g) are the forms $p_1 \in \Lambda^4 M$ and $p_2 \in \Lambda^8 M$ given by

$$p_1 = -\frac{1}{8\pi^2} \text{Tr } R^2 , \quad p_2 = \frac{1}{128\pi^4} ((\text{Tr } R^2)^2 - 2 \text{Tr } R^4) ,$$

where, for any positive integer k , the *trace forms* are

$$\begin{aligned} \text{Tr } R^{2k}(X_1, \dots, X_{4k}) &= \\ \frac{1}{4k!} \sum_{\sigma} \epsilon(\sigma) \text{Tr}(R(X_{\sigma(1)}, X_{\sigma(2)}) \circ \dots \circ R(X_{\sigma(4k-1)}, X_{\sigma(4k)})) , \end{aligned}$$

and the summation is taken over all permutations σ of $\{1, \dots, 4k\}$. The first-order corrections of the Maxwell equations that we are interested in are (see [11] and, e.g., also [12])

$$d \star F + \frac{1}{2} F \wedge F = -\beta p(M, g) \tag{4.6}$$

where β is a real constant and $p(M, g)$ is the 8-form on M given by

$$\begin{aligned} p(M, g) &= 64\pi^4(p_1^2 - 4p_2) \\ &= 4 \operatorname{Tr} R^4 - (\operatorname{Tr} R^2)^2 . \end{aligned}$$

We remark that β is a dimensionful *nonnegative* constant, proportional to the sixth power of the eleven-dimensional Planck length; after a unit of length had been fixed once and for all, it is entirely natural to look for backgrounds which satisfy (4.6) for a definite value of this “parameter”. We will see that this is indeed the case and that, at least for the class of backgrounds considered in this paper, this value is automatically dictated.

To state the main Theorem 4.5 of this section, we need some preliminary notions and results. The *Ricci form* $\rho_1(U_1, U_2) = \operatorname{Ric}_h(JU_1, U_2)$ of a Kähler manifold (B, h, J, ω) is related to the trace of its curvature by (see, e.g., [28, Vol.II]):

$$\rho_1(U_1, U_2) = \frac{1}{2} \operatorname{Tr}(J \circ R_h(U_1, U_2)) ;$$

we similarly define the *second Ricci form* $\rho_2 \in \Lambda^6 B$ by

$$\begin{aligned} \rho_2(U_1, \dots, U_6) &= \\ \frac{1}{6!} \sum_{\sigma} \epsilon(\sigma) \operatorname{Tr}(J \circ R_h(U_{\sigma(1)}, U_{\sigma(2)}) \circ R_h(U_{\sigma(3)}, U_{\sigma(4)}) \circ R_h(U_{\sigma(5)}, U_{\sigma(6)})) , \end{aligned}$$

where the summation is over all permutations σ of $\{1, \dots, 6\}$.

Definition 4.2. A 10-dimensional Kähler manifold is called *admissible* if $\rho_1 \in \Lambda^2 B$, $\operatorname{Tr} R_h^2 \in \Lambda^4 B$, $\rho_2 \in \Lambda^6 B$ and $\operatorname{Tr} R_h^4 \in \Lambda^8 B$ are all zero.

We note that any admissible Kähler manifold is in particular Ricci-flat. The bridge of Sasakian manifolds with M-theory is provided by the following.

Proposition 4.3. *Let M be a Lorentzian Sasakian manifold with a Kähler base B which is admissible. Then $p(M, g) = -\frac{6688}{105} \pi^* \omega^4$.*

This result is a consequence of the fact that the trace forms of a general 11-dimensional Lorentzian Sasakian manifold are

$$\operatorname{Tr} R^2 = \pi^* \left\{ \operatorname{Tr} R_h^2 - \frac{4}{3} \rho_1 \wedge \omega - 8\omega^2 \right\} \quad (4.7)$$

and

$$\begin{aligned} \operatorname{Tr} R^4 &= \pi^* \left\{ \operatorname{Tr} R_h^4 - \frac{2}{7} \rho_2 \wedge \omega - \frac{2}{35} \operatorname{Tr} R_h^2 \wedge \omega^2 \right. \\ &\quad \left. + \frac{8}{315} \rho_1 \wedge \omega^3 + \frac{8}{105} \omega^4 \right\} . \quad (4.8) \end{aligned}$$

These equations are obtained by somewhat long and tedious computations and a repeated use of the algebraic Bianchi identities. We omit the details.

We remark that there does *not* exist any definitive notion of a *super-symmetric* solution of equations (4.1) and (4.6); more precisely there are no complete results for the corrections required at order β to the covariant derivative $\nabla_X^o \varphi := \nabla_X \varphi + i(\frac{1}{6} \iota_X F + \frac{1}{12} X^\flat \wedge F) \cdot \varphi$. As a matter of fact the modifications described so far in the literature had all been obtained

by requiring that supersymmetry is preserved on some particular classes of backgrounds (see [29]; see also [14] and references therein for a more recent discussion on this circle of ideas). In this regard, we also have to note that two first order corrections of ∇^o are usually seen as equivalent if they both act trivially on the same putative parallel spinor φ .

The modified spinorial connection ∇^β is not arbitrary but it has to satisfy two basic properties:

- (i) if $\beta = 0$ then $\nabla^\beta = \nabla^o$, and
- (ii) ∇^β depends just on g and F and not on any geometric datum specific of the backgrounds considered (e.g. ω , ξ or η in our case).

The following proposal satisfies these two properties.

Definition 4.4. We call a solution of (4.1) and (4.6) *supersymmetric* if it has a spin structure and a non-zero pseudo-Majorana spinor parallel with respect to the connection

$$\nabla_X^\beta \varphi := \nabla_X^o \varphi + i\beta \left\{ \mu_1 \iota_X p(M, g) + \mu_2 X^\flat \wedge p(M, g) \right\} \cdot \varphi ,$$

where

$$\mu_1 = 1 - \frac{1+489\sqrt{6}}{1200} \quad \text{and} \quad \mu_2 = \frac{1}{12} \cdot \frac{\sqrt{3}-\sqrt{2}}{(3\sqrt{3}+4\sqrt{2})} .$$

Our main result is then the following

Theorem 4.5. *Any Lorentzian Sasakian manifold M with an admissible base B is a solution of (4.1) and (4.6) with the non-zero flux*

$$F = \sqrt{\frac{3}{2}} \pi^* \omega^2 .$$

If the base B is admissible with full holonomy $\text{SU}(n)$ or it is the standard complex torus with its periodic “Ramond” spin structure then the solution is supersymmetric.

Proof. From $\text{Ric}_h = 0$ and Theorem 4.1 one knows that (4.1) is satisfied with $\lambda^2 = \frac{3}{2}$. By the discussion in §4.1 and Proposition 4.3, equation (4.6) holds if and only if $\lambda > 0$ and

$$\beta = \frac{105}{6688} \left(\sqrt{\frac{2}{3}} + \frac{3}{4} \right) . \tag{4.9}$$

This proves the first part of the theorem, where $\lambda = \sqrt{\frac{3}{2}}$.

If B is the standard complex torus or has holonomy $\text{SU}(n)$ then Theorem 3.2 applies and M has two generalised Killing spinors φ_\pm satisfying (3.7) with $n = 5$, $\epsilon = -1$. We recall that fixing an adapted frame at $x \in M$ as at the end of §1.2 yields appropriate identifications $TM|_x \simeq V = W \oplus \mathbb{R}\xi$ and $\mathbb{S}(M)|_x \simeq \mathbb{S} = \Lambda^\bullet U^*$ in such a way that $\varphi_+ \in \Lambda^0 U^*$ and $\varphi_- \in \Lambda^5 U^*$. In particular,

$$\begin{aligned} e_i \cdot e_{i+5} \cdot \varphi_\pm &= i(e_i^{10} + e_i^{01}) \cdot (e_i^{10} - e_i^{01}) \cdot \varphi_\pm \\ &= -i[e_i^{10}, e_i^{01}] \cdot \varphi_\pm = \pm i\varphi_\pm , \end{aligned}$$

for any $i = 1, \dots, 5$,

$$\pi^* \omega^2 \cdot \varphi_{\pm} = 2 \sum_{1 \leq i < j \leq 5}^5 e_i \cdot e_{i+5} \cdot e_j \cdot e_{j+5} \cdot \varphi = -20\varphi_{\pm} ,$$

and

$$\begin{aligned} \pi^* \omega^4 \cdot \varphi_{\pm} = \\ 24 \sum_{1 \leq i < j < k < l \leq 5}^5 e_i \cdot e_{i+5} \cdot e_j \cdot e_{j+5} \cdot e_k \cdot e_{k+5} \cdot e_l \cdot e_{l+5} \cdot \varphi = 120\varphi_{\pm} . \end{aligned}$$

Our first aim is to show $\nabla_{\xi}^{\beta} \varphi_{\pm} = 0$, where the parameter β is as in (4.9). On the one hand $\nabla_{\xi} \varphi_{\pm} = \pm i \frac{5}{2} \varphi_{\pm}$ and from

$$\iota_{\xi} \pi^* \omega^2 \cdot \varphi_{\pm} = 0 , \quad \text{and} \quad (\xi^{\flat} \wedge \pi^* \omega^2) \cdot \varphi_{\pm} = \xi^{\flat} \cdot \pi^* \omega^2 \cdot \varphi_{\pm} = \mp 20\varphi_{\pm} ,$$

we find that $\nabla_{\xi}^o \varphi_{\pm} = \pm 5i(\frac{1}{2} - \frac{1}{\sqrt{6}})\varphi_{\pm}$. From this fact, Proposition 4.3 and

$$\iota_{\xi} \pi^* \omega^4 \cdot \varphi_{\pm} = 0 , \quad \text{and} \quad (\xi^{\flat} \wedge \pi^* \omega^4) \cdot \varphi_{\pm} = \xi^{\flat} \cdot \pi^* \omega^4 \cdot \varphi_{\pm} = \pm 120\varphi_{\pm} ,$$

one finally gets $\nabla_{\xi}^{\beta} \varphi_{\pm} = 0$.

To prove $\nabla_X^{\beta} \varphi_{\pm} = 0$ for all horizontal vectors, we need few additional identities which hold for any spinor $\varphi \in \Lambda^p U^*$. First a computation similar to (3.6) yields

$$\pi^* \omega \cdot \varphi = \sum_{i=1}^5 e_i \cdot e_{i+5} \cdot \varphi = (5 - 2p)i\varphi . \quad (4.10)$$

Secondly by considering the general relations (see, e.g., [30])

$$(X^{\flat} \wedge \alpha) \cdot \varphi = X^{\flat} \cdot \alpha \cdot \varphi + \iota_X \alpha \cdot \varphi$$

and

$$(X^{\flat} \wedge \alpha) \cdot \varphi = (-1)^{|\alpha|} \alpha \cdot X^{\flat} \cdot \varphi - \iota_X \alpha \cdot \varphi$$

in the case $\alpha = \pi^* \omega \in \Lambda^2 M$ and $X \in \mathfrak{X}(M)$ horizontal, one gets

$$X^{\flat} \cdot \pi^* \omega \cdot \varphi - \pi^* \omega \cdot X^{\flat} \cdot \varphi = -2\iota_X \pi^* \omega \cdot \varphi = 2\Phi(X) \cdot \varphi . \quad (4.11)$$

Using (4.11) and (4.10) one has for any horizontal vector

$$\begin{aligned} \Phi(X) \cdot \varphi_{\pm} &= \frac{1}{2}(X^{\flat} \cdot \pi^* \omega - \pi^* \omega \cdot X^{\flat}) \cdot \varphi_{\pm} \\ &= \frac{1}{2}i \cdot (\pm 2)X^{\flat} \cdot \varphi_{\pm} \\ &= \pm iX^{\flat} \cdot \varphi_{\pm} \end{aligned} \quad (4.12)$$

and since φ_{\pm} is generalised Killing,

$$\begin{aligned} \nabla_X \varphi_{\pm} &= -\frac{1}{2}\Phi(X) \cdot \xi \cdot \varphi_{\pm} = \mp \frac{1}{2}\Phi(X) \cdot \varphi_{\pm} \\ &= -\frac{1}{2}iX^{\flat} \cdot \varphi_{\pm} . \end{aligned}$$

A direct computation together with (4.12) implies also

$$\begin{aligned} \iota_X \pi^* \omega^2 \cdot \varphi_{\pm} &= \mp 8i \Phi(X) \cdot \varphi_{\pm} = 8X^{\flat} \cdot \varphi_{\pm} , \\ (X^{\flat} \wedge \pi^* \omega^2) \cdot \varphi_{\pm} &= X^{\flat} \cdot \pi^* \omega^2 \cdot \varphi_{\pm} + \iota_X \pi^* \omega^2 \cdot \varphi_{\pm} = -12X^{\flat} \cdot \varphi_{\pm} , \end{aligned}$$

and finally $\nabla_X^o \varphi_{\pm} = i \frac{\sqrt{2}-\sqrt{3}}{2\sqrt{3}} X^{\flat} \cdot \varphi_{\pm}$. From this fact, Proposition 4.3 and

$$\begin{aligned} \iota_X \pi^* \omega^4 \cdot \varphi_{\pm} &= \pm 96i \Phi(X) \cdot \varphi_{\pm} = -96X^{\flat} \cdot \varphi_{\pm} , \\ (X^{\flat} \wedge \pi^* \omega^4) \cdot \varphi_{\pm} &= X^{\flat} \cdot \pi^* \omega^4 \cdot \varphi_{\pm} + \iota_X \pi^* \omega^4 \cdot \varphi_{\pm} = 24X^{\flat} \cdot \varphi_{\pm} , \end{aligned}$$

one gets $\nabla_X^{\beta} \varphi_{\pm} = 0$ for all horizontal vectors too.

We have seen $\nabla^{\beta} \varphi_{\pm} = 0$. To show that the solution is supersymmetric one has simply to note that the reality condition $j(\varphi) = \varphi$ is satisfied for an appropriate linear combination $\varphi = c_+ \varphi_+ + c_- \varphi_-$ with constant coefficients (recall the description of the pseudo-Majorana conjugation given in §4.1) and that such combination φ is still ∇^{β} -parallel. The proof is completed. \square

We now comment on the class of admissible Kähler manifolds. We note that it is not empty as it includes all flat Kähler manifolds B ; the corresponding Lorentzian Sasakian manifolds given by the total spaces of the S^1 -bundles $\pi : M \rightarrow B$ provide new non-flat M-theory backgrounds with a nonzero flux F . For instance in the special case of the standard complex torus the Kähler form is integral and therefore M is globally defined and compact: it is, in fact, the compact quotient of the 11-dimensional simply connected real Heisenberg group, see e.g. [31] for its explicit description. By Theorem 4.5 this solution is also supersymmetric.

It is a natural problem to understand whether admissible non-flat Kähler manifolds do actually exist. Slightly more generally one might also note that the right hand side of equation (4.6) is given by an exact form, due to our ansatz (4.3), and then consider 10-dimensional Ricci-flat Kähler manifolds for which $\text{Tr } R_h^2 \in \Lambda^4 B$, $\rho_2 \in \Lambda^6 B$ and $\text{Tr } R_h^4 \in \Lambda^8 B$ are all constant multiples of the appropriate powers of the Kähler form (such manifolds too would determine solutions of (4.1) and (4.6), as it easily follows from equations (4.7) and (4.8)).

In this regard we stress that $\text{Tr } R_h^2$, ρ_2 and $\text{Tr } R_h^4$ are all closed and of type (p, p) . It follows from Hodge theory and Serre duality that the above conditions are satisfied up to exact terms on compact Calabi-Yau 5-folds with Hodge numbers $h^{1,1} = h^{2,2} = 1$. By a deep result of [13] all complete intersection Calabi-Yau 5-folds which can be defined in a single projective space are of this type (see [32] for the definition and basic properties; see also the list given in [13, Table 5]). To get further insight on these manifolds seems like an extremely difficult task [33].

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